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Engineering Notes

Minimum-Fuel Cruise at Constant Altitude with Fixed Arrival Time

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Introduction

An IMPORTANT problem in air traffic management (ATM) is the design of aircraft trajectories that meet certain arrival time constraints at given waypoints, for instance, at the top of descent (TOD), at the initial approach fix (IAF), or at the runway threshold. These are called 4-D trajectories, which are a key element in the trajectory-based-operations (TBO) concept proposed by NextGen and SESAR for the future ATM system. The time constraints must also be met in certain cases in which the nominal trajectories have to be modified to resolve detected conflicts (e.g., lost of separation minima) between aircraft; for example, Bilimoria and Lee [1] analyze aircraft conflict resolution with an arrival time constraint at a downstream waypoint.

On the other hand, the design of fuel-optimal trajectories that lead to energy-efficient flights is another important problem which has been treated extensively in the literature, see, for instance, Burrows [2], Neuman and Kreindler [3], Menon [4] and the references therein. Fuel-optimal trajectories with fixed arrival times are studied by Sorensen and Waters [5], Burrows [6] and Chakravarty [7], who analyze the 4-D fuel-optimization problem as a minimum direct-operating-cost (DOC) problem with free final time, that is, the problem is to find the time cost for which the corresponding free final-time DOC-optimal trajectory arrives at the assigned time.

In this work we analyze the problem of minimum-fuel cruise at constant altitude with a fixed arrival time as a singular optimal control problem, building upon the works of Pargett and Ardema [8] and Rivas and Valenzuela [9], who analyze the problem of maximum-range cruise at constant altitude also as a singular optimal control problem; the case of unsteady cruise is considered. The singular arcs and the corresponding optimal control are obtained as a function of the final time. The optimal paths are obtained as well, which define a variable-Mach cruise at constant altitude. The influence of cruise altitude on the optimal paths is analyzed, and the minimum fuel is calculated. The final-time constraint may be defined, for example, by a flight delay imposed on the nominal (preferred) cruise trajectory (which in our case is the minimum-fuel cruise trajectory with free final time); comparison with a standard constant-Mach procedure to absorb delays is made. Results are presented for a model of a Boeing 767-300ER.

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Problem Formulation

It is desired to minimize fuel consumption for a given range and a given final time, that is, to minimize the following performance index:

$$J = \int_0^{t_f} cT\,\mathrm{d}t \tag{1}$$

with final time t_f fixed, subject to the following constraints:

$$\dot{V} = \frac{1}{m}(T - D) \qquad \dot{m} = -cT \qquad \dot{x} = V \tag{2}$$

which are the equations of motion for cruise at constant altitude and constant heading. In these equations, the drag is a general known function $D(V, m)$, which takes into account the remaining equation of motion $L = mg$. The thrust $T(V)$ is given by $T = \pi T_M(V)$, where π models the throttle setting $\pi_m \leq \pi \leq 1$, and $T_M(V)$ is a known function. The specific fuel consumption $c(V)$ is also a known function. Thus, this problem has three states (speed V, mass m, and flight distance x) and one control (π). The initial aircraft mass (m_i) and the initial and final flight distances ($x_i = 0$ and x_f) are given.

To solve this problem it is convenient to take the distance x as the independent variable. Thus, the problem can be formulated as follows: minimize the performance index

$$J = \int_0^{x_f} \frac{c\pi T_M}{V}\mathrm{d}x \tag{3}$$

with final distance x_f fixed, subject to the following constraints (the prime denotes derivation with respect to x):

$$V' = \frac{\pi T_M - D}{mV} \qquad m' = -\frac{c\pi T_M}{V} \qquad t' = \frac{1}{V} \tag{4}$$

with final time t_f fixed (this final-time constraint is a constraint in a state variable [10]).

The Hamiltonian of this problem is given by

$$H = \frac{c\pi T_M}{V} + \lambda_V \frac{\pi T_M - D}{mV} - \lambda_m \frac{c\pi T_M}{V} + \frac{\lambda_t}{V} \tag{5}$$

where λ_V, λ_m, and λ_t are the adjoint variables, which are defined by the following equations [10]:

$$\begin{aligned}
\lambda_V' &= -\frac{\partial H}{\partial V} = \frac{\lambda_V}{mV}\left(\frac{\partial D}{\partial V} - \frac{D}{V}\right) + \frac{\lambda_t}{V^2} \\
&\quad - \left[\frac{\lambda_V}{m} - (\lambda_m - 1)c\right]\left(\frac{1}{V}\frac{\mathrm{d}T_M}{\mathrm{d}V} - \frac{T_M}{V^2}\right)\pi \\
&\quad + (\lambda_m - 1)\frac{\mathrm{d}c}{\mathrm{d}V}\frac{T_M}{V}\pi \\
\lambda_m' &= -\frac{\partial H}{\partial m} = \frac{\lambda_V}{mV}\left(\frac{\partial D}{\partial m} - \frac{D}{m} + \frac{T_M}{m}\pi\right) \qquad \lambda_t' = -\frac{\partial H}{\partial t} = 0
\end{aligned} \tag{6}$$

Note that from the third equation one gets $\lambda_t =$ constant.

Because H is linear in the control variable one can write $H = \bar{H} + S\pi$, with

$$\bar{H} = \left(\lambda_t - \lambda_V \frac{D}{m}\right)\frac{1}{V} \qquad S = \left[\frac{\lambda_V}{m} - (\lambda_m - 1)c\right]\frac{T_M}{V} \tag{7}$$

The function S is called the switching function. The necessary conditions for optimality are stated in [10].

Singular Arc

Because H is not an explicit function of distance, one has $H =$ constant on the optimal trajectory. The singular control is obtained when the switching function is zero ($S = 0$) on an interval of distance; hence, because $H =$ constant, one also has $\bar{H} =$ constant. On that interval of distance one has $S' = 0$ as well. The singular arc is defined by the three equations: $\bar{H} =$ constant, $S = 0$, $S' = 0$ (see [10]).

Taking into account the state Eq. (4) and the adjoint Eq. (6), the function S' is given by

$$S' = -\left[\frac{\lambda_V}{m} - (\lambda_m - 1)c\right]\frac{D}{mV^2}\left(\frac{\mathrm{d}T_M}{\mathrm{d}V} - \frac{T_M}{V}\right) + \frac{T_M}{mV^2}\left[\frac{\lambda_V}{m}\left(\frac{\partial D}{\partial V} - \frac{D}{V} + cD - mc\frac{\partial D}{\partial m}\right) + \frac{\lambda_t}{V} + (\lambda_m - 1)D\frac{\mathrm{d}c}{\mathrm{d}V}\right] \tag{8}$$

Note that the terms in the control variable π have cancelled out of this equation.

Hence, the three equations that define the singular arc ($\bar{H}=$ constant, $S = 0$, $S' = 0$) reduce to

$$\left(\lambda_t - \lambda_V\frac{D}{m}\right)\frac{1}{V} = H \qquad \frac{\lambda_V}{m} - (\lambda_m - 1)c = 0$$
$$\frac{\lambda_V}{m}\left(\frac{\partial D}{\partial V} - \frac{D}{V} + cD - mc\frac{\partial D}{\partial m}\right) + \frac{\lambda_t}{V} + (\lambda_m - 1)D\frac{\mathrm{d}c}{\mathrm{d}V} = 0 \tag{9}$$

where H is the constant value of the Hamiltonian.

Eliminating the adjoint variables λ_V and λ_m from these equations and making $\omega_t = \frac{\lambda_t}{H}$, one obtains the following expression

$$D\left[\left(1 - \frac{\omega_t}{\omega_t - V}\right) - Vc - \frac{V}{c}\frac{\mathrm{d}c}{\mathrm{d}V}\right] - V\frac{\partial D}{\partial V} + Vcm\frac{\partial D}{\partial m} = 0 \tag{10}$$

which is a family of singular arcs of parameter ω_t. The value of this constant is determined by the constraint $t(x_f) = t_f$. The corresponding singular arc is the solution to the problem. The problem of free final time corresponds to the case $\omega_t = 0$; in this case, one has the same singular arc obtained in [9] for the problem of maximum-range cruise for a given fuel load.

Equation (10) defines a family of singular lines in the (V, m) space, which are in fact, the loci of possible points in which optimal paths can lie.

Dimensionless Singular Arc

Let us consider a general drag-polar $C_D = C_D(C_L, M)$, where C_D and C_L are the drag and lift coefficients, and $M = V/a$ is the Mach number (a being the speed of sound). From the definition of C_D and C_L one has $D = \frac{1}{2}\rho V^2 S_W C_D(C_L, M)$ and $C_L = \frac{mg}{\frac{1}{2}\rho V^2 S_W}$, where the equation of motion $L = mg$ has been taken into account (ρ is the air density and S_W the wing surface area). The following partial derivatives are obtained:

$$\frac{\partial D}{\partial V} = \frac{1}{2}\rho V S_W\left(2C_D - 2C_L\frac{\partial C_D}{\partial C_L} + M\frac{\partial C_D}{\partial M}\right) \qquad \frac{\partial D}{\partial m} = g\frac{\partial C_D}{\partial C_L} \tag{11}$$

Also, let us consider a specific fuel consumption coefficient $C = C(M)$, defined by $c = \frac{a}{L_H}C$, where L_H is the fuel latent heat (the dependence of C with the thrust coefficient is neglected [9]). The following derivative is obtained

$$\frac{\mathrm{d}c}{\mathrm{d}V} = \frac{1}{L_H}\frac{\mathrm{d}C}{\mathrm{d}M} \tag{12}$$

Substituting the preceding derivatives into Eq. (10), one gets

$$C_D\left(1 + \frac{\omega_t}{\omega_t - aM} + \frac{a^2}{L_H}MC + \frac{M}{C}\frac{\mathrm{d}C}{\mathrm{d}M}\right) - \left(2 + \frac{a^2}{L_H}MC\right)C_L\frac{\partial C_D}{\partial C_L} + M\frac{\partial C_D}{\partial M} = 0 \tag{13}$$

which is the general dimensionless expression for the family of singular arcs.

Optimal Singular Control

The function S'' depends linearly on the control variable, say $S'' = A(V, m) + B(V, m)\pi$, therefore, because one also has $S'' = 0$ (where $S = 0$), the singular control is obtained from $A(V, m)+ B(V, m)\pi = 0$; after laborious manipulations one gets the following:

$$\pi = \frac{D}{T_M}\left(1 + Vc\frac{A_1(V, m)}{B_1(V, m)}\right) \tag{14}$$

where $A_1(V, m)$ and $B_1(V, m)$ are given by

$$A_1(V, m) = m\frac{\partial^2 D}{\partial m \partial V} - m^2 c\frac{\partial^2 D}{\partial m^2} - mc\frac{\partial D}{\partial m} - \frac{m}{D}\left(\frac{\partial D}{\partial V} - mc\frac{\partial D}{\partial m}\right)$$
$$B_1(V, m) = DV\left(c^2 + 3\frac{\mathrm{d}c}{\mathrm{d}V} + \frac{1}{c}\frac{\mathrm{d}^2 c}{\mathrm{d}V^2}\right) + 2\frac{\partial D}{\partial V}\left(Vc + \frac{V}{c}\frac{\mathrm{d}c}{\mathrm{d}V}\right) - mV\left(c^2 + 3\frac{\mathrm{d}c}{\mathrm{d}V}\right)\frac{\partial D}{\partial m} + V\frac{\partial^2 D}{\partial V^2} + m^2c^2V\frac{\partial^2 D}{\partial m^2} - 2Vcm\frac{\partial^2 D}{\partial m \partial V} \tag{15}$$

This expression for the optimal singular control depends implicitly on the parameter of the family of singular arcs, since V and m are related by the singular arc Eq. (10) which includes the dependence on ω_t.

As indicated in [8], in general one has $Vc \ll 1$, in which case Eq. (14) reduces to $\pi = \frac{D}{T_M}$, that is, one has the well-known cruise hypothesis $T = D$.

The generalized Legendre–Clebsch condition (see [11]) establishes that for the singular control to be optimal one must have $-(\partial S''/\partial\pi) \geq 0$, that is, one must have the following necessary condition:

$$B(V, m) \leq 0 \tag{16}$$

The function $B(V, m)$ is given by

$$B(V, m) = \frac{T_M^2}{m^3 V^4}\lambda_V B_1(V, m) \tag{17}$$

and the necessary condition (16) can be shown to be satisfied numerically for the aircraft model defined in the following section; in fact, it can be shown that $\lambda_V < 0$ and $B_1 > 0$.

Optimal Paths

In this work we focus on the analysis of the singular arcs, and thus we consider optimal paths composed solely of a singular arc; the analysis of these kinds of optimal trajectories implicitly assumes that the initial and final points of the trajectory belong to the singular arc. In a more general case, with arbitrarily given initial and final velocities, the optimal trajectory will be composed of a minimum/maximum-thrust arc, a singular arc, and a final minimum/maximum-thrust arc (minimum or maximum depending on the given velocities), with an optimal control of the type bang-singular-bang, as described by Bryson and Ho [10].

Results

The aircraft model considered in this paper for the numerical applications is that of a Boeing 767-300ER (a typical twin-engine,

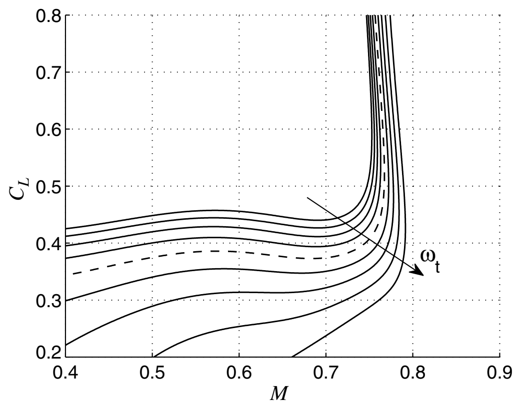

Fig. 1 Singular arcs in the $C_L - M$ plane for $h_A = 10000$ m ($\omega_t = -100, -75, -50, -25, 0, 25, 50, 75, 100$ m/s; dashed line corresponds to $\omega_t = 0$).

wide-body, long-range transport aircraft) which is described in [9], and summarized in the Appendix.

Only one value of cruise range is considered: $x_f = 10000$ km, and the initial weight at the start of the cruise is the same in all cases: $W_i = 1600$ kN. The given cruise altitude is denoted as h_A.

In the computation, when ω_t is given the initial speed $V(0)$ is defined by Eq. (10), and the optimal control π is given by Eq. (14), hence, the state Eq. (4) are integrated (using MATLAB's ode45) until $x = x_f$. However, when t_f is given, for a chosen value of $V(0)$ and for the optimal control π given by Eq. (14), the state Eq.. (4) is integrated (using MATLAB's ode45) until $x = x_f$; then one checks whether the corresponding final time is t_f or not: if not, a new value of $V(0)$ is chosen and the procedure is repeated until $t(x_f) = t_f$ [the corresponding value of ω_t is then defined by Eq. (10)].

Singular Arc

The singular arcs in the $C_L - M$ plane defined by Eq. (13) are plotted in Fig. 1 for different values of the parameter ω_t and for a representative altitude $h_A = 10000$ m. The curve for $\omega_t = 0$ corresponds to the problem of free final time. Note that these curves present a maximum value of the Mach number, as discussed in [9].

Regarding the effect of the cruise altitude in these dimensionless arcs, it can be shown that it is very small, with no visible effect on Fig. 1 (it was shown to be negligible in [9] in the case $\omega_t = 0$).

Optimal Paths

From the definition of C_L one has

$$\frac{1}{2}\gamma p_A S_W M^2 C_L(M) = W \tag{18}$$

where W is the aircraft weight, γ the ratio of specific heats, and p_A the pressure at the given altitude (h_A). This expression defines the family of singular arcs in the $W - M$ plane for each value of h_A (it depends implicitly on ω_t).

The optimal paths are represented in Fig. 2 for different values of cruise altitude ($h_A = 9000, 10000, 11000$, and 12000 m); the optimal paths are represented by thick lines superposed on the singular arcs. In each graph, optimal paths for different values of the parameter ω_t are plotted (the relationship between ω_t and t_f is described below). Note that depending on the cruise altitude the optimal procedure changes: for a given value of ω_t, at small altitudes the speed decreases as fuel is consumed, whereas at large altitudes the speed increases

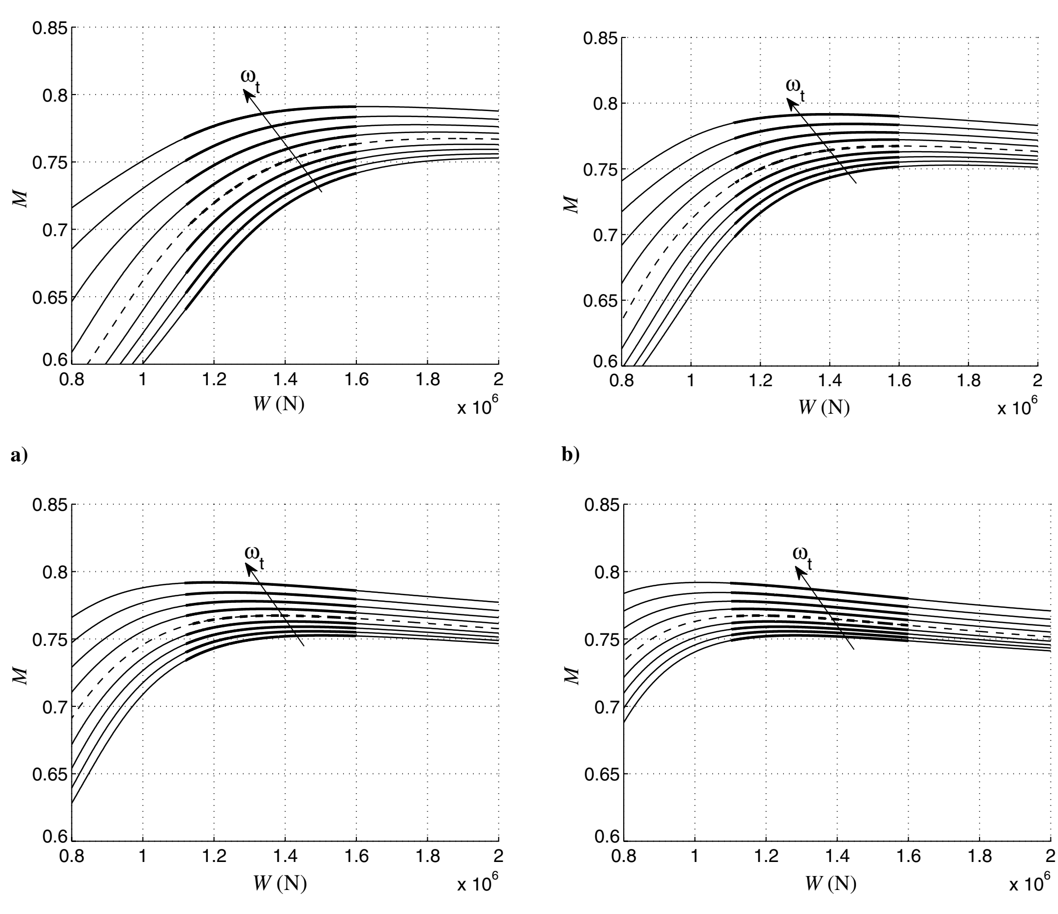

Fig. 2 Optimal paths for various values of h_A: a) 9000, b) 10,000, c) 11,000, d) 12,000 m ($\omega_t = -100, -75, -50, -25, 0, 25, 50, 75, 100$ m/s; dashed lines corresponds to $\omega_t = 0$).

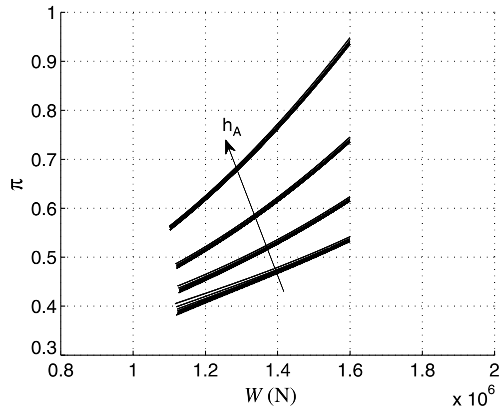

Fig. 3 Optimal singular control for $h_A = 9000$; 10,000; 11,000; 12,000 m ($\omega_t = -100, -75, -50, -25, 0, 25, 50, 75, 100$ m/s).

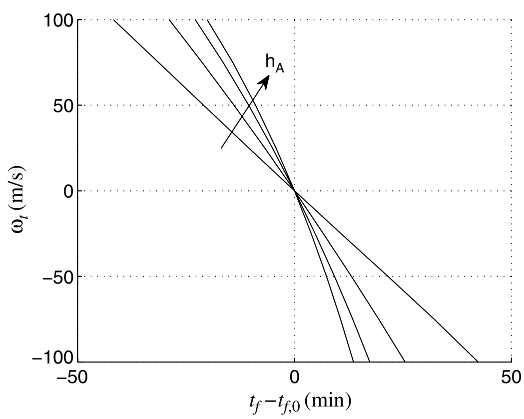

Fig. 4 Parameter ω_t vs flight time ($h_A = 9000$; 10,000; 11,000; 12,000 m).

(although slightly), as analyzed in [9]; also, at certain altitudes and values of ω_t, the optimal Mach number is roughly constant.

Implicit in the results presented in Fig. 2 is that the entire cruise can follow the singular arc. The inequality constraint $\pi_m \leq \pi \leq 1$ is satisfied for those optimal paths. The optimal singular control for the optimal paths represented in Fig. 2 is depicted in Fig. 3. Note that the optimal control decreases as fuel is consumed and increases as cruise altitude increases.

For a given ω_t, the computation of each optimal trajectory ends when the fixed distance $x_f = 10000$ km is reached; the final optimal values of fuel consumption m_F and flight time t_f are then obtained. The relationship between ω_t and t_f is represented in Fig. 4, and some numerical values are collected in Table 1; in the figure $t_{f,0}$ is the nominal flight time in the problem of free final time that corresponds to $\omega_t = 0$ (note that $t_{f,0}$ depends on cruise altitude). One has that positive values of ω_t correspond to flight times smaller than $t_{f,0}$; in this case the optimal procedure requires larger speeds (see Fig. 2).

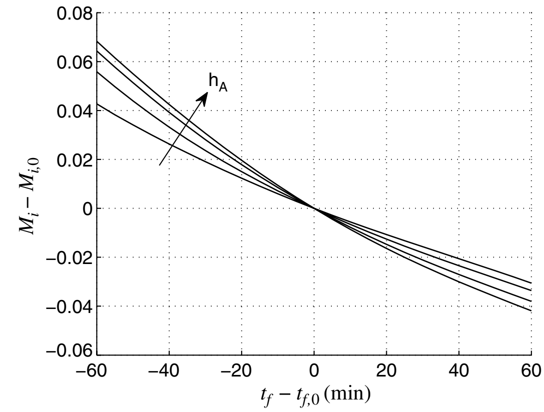

Fig. 5 Optimum initial Mach number vs flight time ($h_A = 9000$; 10,000; 11,000; 12,000 m).

On the other hand, negative values of ω_t correspond to flight times larger than $t_{f,0}$; now optimal paths require smaller speeds (see Fig. 2).

Of particular interest in the problem is the optimal initial Mach number M_i [which is defined for each value of ω_t by Eq. (10)]; from the computational point of view, M_i allows one to integrate and compute the optimal path. If one lets $M_{i,0}$ be the initial Mach number in the nominal free final-time problem ($\omega_t = 0$), the increment $M_i - M_{i,0}$ is represented in Fig. 5 as a function of the increment in flight time $t_f - t_{f,0}$ for different cruise altitudes (note that $M_{i,0}$ depends on cruise altitude). As indicated earlier, larger flight times require smaller Mach numbers and vice versa. One also has that for flight times larger than $t_{f,0}$, the larger the cruise altitude, the smaller the increment $M_i - M_{i,0}$, and for flight times smaller than $t_{f,0}$, the larger the cruise altitude, the larger $M_i - M_{i,0}$.

Minimum Fuel Consumption

For each value of the flight time t_f, the corresponding optimal path leads to minimum fuel consumption m_F. Let $m_{F,0}$ be the nominal minimum fuel consumption in the problem of free final time (note that $m_{F,0}$ varies with cruise altitude, see Table 1). The increment in fuel consumption $m_F - m_{F,0}$ is represented in Fig. 6 as a function of the increment in flight time $t_f - t_{f,0}$ for different cruise altitudes, and some numerical values are collected in Table 1. One can see that fuel consumption always increases, both when the flight time is larger and when it is smaller than the nominal flight time $t_{f,0}$. The increment in fuel consumption increases with altitude, quite strongly in the case $t_f < t_{f,0}$ and moderately in the case $t_f > t_{f,0}$.

Application: Flight Delay

In actual operational practice an aircraft may be required to absorb a given flight delay along the cruise (reaching the TOD point at a given time) or a given advance (which may be required, for instance, to resolve a conflict). Delays correspond to positive values of $t_f - t_{f,0}$, and advances correspond to negative values.

Table 1 Final optimal values of fuel consumption m_F and flight time t_f

h_A, m	9,000	9,000	10,000	10,000	11,000	11,000	12,000	12,000
ω_t, m/s	m_F, kg	t_f, h	m_F, kg	t_f, h	m_F, kg	t_f, h	m_F, kg	t_f, h
100.0	49,407	11.6686	48,682	11.7411	49,225	11.9139	51,065	11.9675
75.0	49,024	11.8420	48,414	11.8678	49,005	12.0209	50,864	12.0663
50.0	48,790	12.0151	48,255	11.9901	48,881	12.1192	50,755	12.1539
25.0	48,665	12.1887	48,173	12.1082	48,820	12.2103	50,703	12.2324
0	48,624	12.3644	48,148	12.2225	48,802	12.2951	50,688	12.3033
−25.0	48,650	12.5431	48,163	12.3334	48,813	12.3743	50,698	12.3677
−50.0	48,731	12.7234	48,210	12.4412	48,845	12.4487	50,723	12.4264
−75.0	48,855	12.9009	48,281	12.5460	48,892	12.5187	50,759	12.4803
−100.0	49,008	13.0704	48,371	12.6476	48,950	12.5846	50,803	12.5299

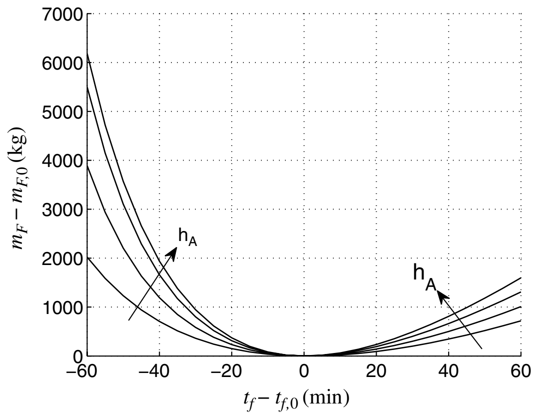

Fig. 6 minimum fuel consumption vs flight time ($h_A = 9000$; 10,000; 11,000; 12,000 m).

Let Δt_f be the delay/advance time (when it is positive it is a delay and when it is negative it represents an advance), then one has $t_f = t_{f,0} + \Delta t_f$ (the nominal free final-time problem corresponds then to $\Delta t_f = 0$). Figure 6 clearly indicates that fuel consumption always increases, both for advances and for delays; for the same delay/advance Δt_f, the increase in the case of advances is much larger than in the case of delays. The increment in fuel consumption increases with altitude.

Thus, one can conclude that departing from the nominal optimal path always has an extra cost, which increases as the delay/advance time increases and as the cruise altitude increases.

Comparison with a Standard Constant-Mach Procedure

Optimal solutions are useful to quantify the nearness to optimality of current-practice procedures. In this section we compare the optimal solution obtained in this paper with the standard procedure to absorb delays consisting in cruising at constant Mach, at the precise Mach value that meets the specified arrival time ($t_f = t_{f,0} + \Delta t_f$), which is trivially obtained because the cruise distance and the flight time are given.

The fuel consumption for the standard procedure is represented in Fig. 7 for two representative altitudes, 9000 and 11000 m, along with the corresponding optimal results (which are also represented in Fig. 6). The results show that the optimal variable-Mach procedure is better than the standard one, especially at small altitudes, in which, as shown in Fig. 2a, the optimal speed law defines a strong variation of Mach number with weight so that the difference with the constant-Mach procedure is large. On the other hand, at large altitudes the optimal speed law defines a weak variation of Mach number with weight (see Fig. 2c), and the difference between the optimal and the standard cases is smaller. At 9000 m the optimal procedure saves between 194.3 and 317.4 kg of fuel, depending on the delay time (from 0 to 60 minutes); and at 11000 m, it saves between 73.3 and 152.3 kg. These values represent savings between 0.40 and 0.64% at 9000 m, and between 0.15 and 0.30% at 11000 m. Finally, note that the larger the delay, the larger the savings.

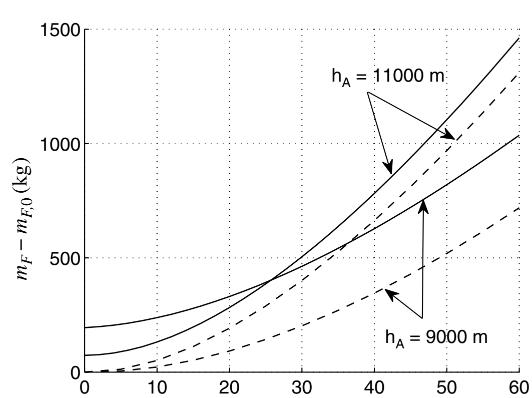

Fig. 7 Fuel consumption vs delay time: a) thick line, standard procedure, b) dashed line, optimal procedure.

Conclusions

Minimum-fuel cruise at constant altitude with a final-time constraint has been analyzed as a singular optimal control problem. The singular arcs and the corresponding optimal paths have been studied as a function of flight time and altitude. The optimal speed law shows, as expected, that longer flight times require smaller Mach numbers and vice versa; in this work, we have been able to quantify this effect, including the influence of cruise altitude.

The minimum fuel consumption has been obtained. It has been shown that the fuel consumption is always larger than that of the nominal free-time problem, both in the case of larger and in the case of smaller flight times. These results allow one to quantify the cost of absorbing flight delays/advances. For the same delay/advance time, advances cost more than delays. The influence of cruise altitude has been also analyzed: the extra cost of absorbing the delays/advances increases with altitude.

From the operational point of view, the optimal variable-Mach solution obtained in the paper presents the drawback of its flyability. However, it can be used to define a flyable procedure close to optimal. This improved procedure would be to cruise flying constant-Mach segments, defined so as to approximate the theoretical variable-Mach curve, that is, a stepped Mach cruise (similarly to the way the stepped cruise climb approximates the optimal cruise climb solution).

Comparison with a standard constant-Mach procedure to absorb delays has been made. It has been shown that the optimal procedure analyzed in the paper gives better performance in terms of fuel savings, especially at low altitudes in which the optimal speed law defines a strong variation of Mach number with weight. At high altitudes, however, the variation of Mach number with weight is weak, and the performance improvement of the optimal procedure is somewhat smaller.

The analysis made allows one to plan the optimal procedure to absorb a given delay along the cruise. This analysis can be also used when the aircraft is informed that a delay is required while flying the cruise. In this case, the optimal path must be preceded by a horizontal deceleration segment flown with idle engine (this corresponds to a control of the type bang-singular).

In this work it has been assumed that the optimal procedure to meet the final-time constraint is performed at the original (nominal) cruise altitude. However, one might consider the problem of meeting the final-time constraint including changes in altitude; this study is left for future work. Also for future work we leave the analysis of the effects of real atmosphere conditions and an average cruise wind.

Appendix: Aircraft Model

The aircraft model of the Boeing 767-300ER is described in more detail in [9], in which it is shown that it provides accurate results. The model has wing surface area 283.3 m^2, maximum take-off mass 186880 kg, and maximum fuel mass 73635 kg. The drag-polar is given by

$$C_D = \left(C_{D_{0,i}} + \sum_{j=1}^{5} k_{0j} K^j(M)\right) + \left(C_{D_{1,i}} + \sum_{j=1}^{5} k_{1j} K^j(M)\right) C_L + \left(C_{D_{2,i}} + \sum_{j=1}^{5} k_{2j} K^j(M)\right) C_L^2 \tag{A1}$$

where

$$K(M) = \frac{(M - 0.4)^2}{\sqrt{1 - M^2}} \tag{A2}$$

Table A1 Compressible drag-polar coefficients for the model aircraft

j	1	2	3	4	5
k_{0j}	0.0067	−0.1861	2.2420	−6.4350	6.3428
k_{1j}	0.0962	−0.7602	−1.2870	3.7925	−2.7672
k_{2j}	−0.1317	1.3427	−1.2839	5.0164	0.0000

The incompressible drag-polar coefficients are $C_{D_{0,i}} = 0.01322$, $C_{D_{1,i}} = -0.00610$, $C_{D_{2,i}} = 0.06000$, and the compressible coefficients are given in Table A1.

The specific fuel consumption coefficient is given by

$$C = c_{\mathrm{SL}} \frac{L_H}{a_{\mathrm{SL}}} (1.0 + 1.2M) \tag{A3}$$

The specific fuel consumption at sea level and $M = 0$ is $c_{\mathrm{SL}} = 9.0 \times 10^{-6}\ \mathrm{kg/(sN)}$ and the fuel latent heat is $L_H = 43 \times 10^6\ \mathrm{J/kg}$.

The maximum thrust is defined by

$$T_M = W_{\mathrm{TO}} \delta C_T \tag{A4}$$

where W_{TO} is the reference takeoff weight, $\delta = p/p_{\mathrm{SL}}$ is the pressure ratio (p_{SL} being the reference sea level pressure), and the thrust coefficient is given by

$$C_T = \frac{T_{\mathrm{SL}}}{W_{\mathrm{TO}}} \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}} (1 - 0.49\sqrt{M}) \frac{1}{\theta} \tag{A5}$$

where $\gamma = 1.4$, $\theta = \Theta/\Theta_{\mathrm{SL}}$ is the temperature ratio (Θ_{SL} being the reference sea-level temperature), and the maximum thrust at sea level and for $M = 0$ is $T_{\mathrm{SL}} = 5.0 \times 10^5$ N.

With respect to the atmosphere, the ISA model is considered.

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